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Nonlinear TE Electromagnetic Surface Waves in a Ferrite Layered Structure

الموجات السطحية الكهرومغناطسية غير الخطية في تركيب طبقي يحتوي على الفرايت

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Abstract

Characteristics of TE electromagnetic surface waves propagating in a nonlinear dielectric film bounded by a ferrite cover are examined theoretically. A dispersion relation based on Jacobian Elliptic Functions is derived, which describes the behaviour of the nonreciprocal nonlinear waves.

Keywords: Nonlinear surface waves, ferrite, dispersion relation.

ملخص

يتناول البحث دراسة نظرية الخصائص انتشار الموجات الكهرومغناطسية (TE) في تركيب يحتوى على شفافية من مادة عازلة غير خطية محاطة بمادة الفرايت. ولقد اشتقت معادلة التشتت التي وجدت أنها تعتمد على دوال جاكوبي (Jacobian Function) التي بدورها وصفت موجات غير خطية وغير متماثلة.

1. Introduction

In several years, the investigation of nonlinear electromagnetic surface and guided waves in nonlinear wave guide structures has been advanced rapidly by many authors (1-10). Recently, the propagation characteristics of nonlinear electromagnetic and magnetostatic surface waves in gyromagnetic (Ferrite materials) wave guide structure have been studied (10-17) as the microwave devices using ferrite materials have unique characteristics; non-reciprocity, and magnetic tenability (17-19). Potential applications of nonlinear electromagnetic waves in designing microwave solid state devices could be utilized by stimulating the study of dispersion relation and the power in a proposed ferrite layered

structure. In this paper, we investigate theoretically the bahaviour of TE surface waves in a nonlinear dielectric film bounded by a gyromagnetic ferrite cover and a linear dielectric substrate. We derive an exact analytical dispersion equation. The layout of the paper is as follows; section two presents the basic equations of electric and magnetic fields components in each layer of wave guide structure. In section 3, we derive the dispersion equations. In section 4, we compute, illustrate, discuss some numerical results and present our final conclusions.

2. Theory

The analytical model of wave-guide structure is shown in Fig.1. A static biasing magnetic field H_0 is applied in the +y direction. We assume that the finite nonlinear film occupies the region $0 \le z \le d$ bounded by the ferrite (YIG) cover of the space $z \ge d$ and a semi-infinite linear dielectric substrate in the region z<0.

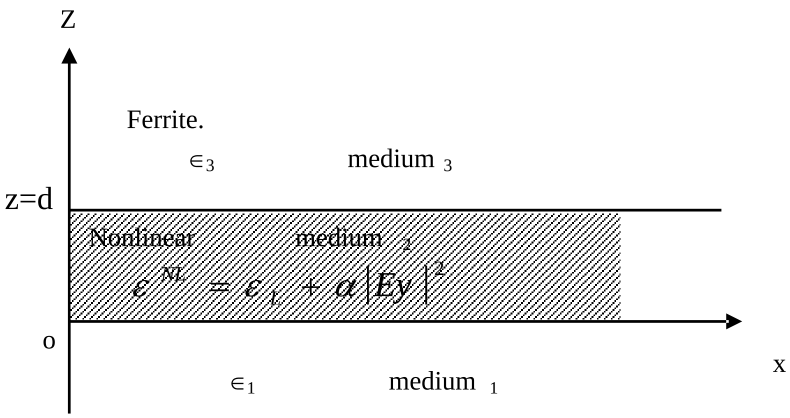


Figure (1): Analytical model of a layered structure and coordinate system

The magnetic permeability tensor of the gyromagnetic ferrite cover is described as (11-14).

$$\mu(\omega) = \begin{bmatrix} \mu_{xx} & 0 & \mu_{xz} \\ 0 & \mu_B & 0 \\ -\mu_{xz} & 0 & -\mu_{xx} \end{bmatrix} \quad (1a)$$

Where

$\mu_{xx} = \mu_B \left(\dfrac{\omega_0(\omega_0 + \omega_m) - \omega^2}{\omega_0^2 - \omega^2} \right)$, $\mu_{xz} = i\mu_B \dfrac{\omega\omega_m}{\omega_0^2 - \omega^2}$, ω is the angular frequency of the supported wave, $\omega_0 = \gamma\mu_0 \mathbf{H}_0, \omega_m = \gamma\mu_0 M_0, \mu_0 \mathbf{H}_0$ is the applied magnetic field, $\gamma = 1.76 \times 10^{11} s^{-1} T^{-1}$ is the gyromagnetic ratio, $\mu_0 M_0$ is the dc saturation magnetization of the magnetic insulator and μ_B has been introduced as the background, optical magnon permeability. It will be assumed that ferrite cover has frequency-independent dielectric constant $\varepsilon = \varepsilon_1$ and has value 1. The dielectric function of the nonlinear dielectric medium is assumed to be Kerr-like, isotropic and depends on the electric field, then the dielectric function of a nonlinear film is characterized by (1-11):

$$\varepsilon^{NL} = \varepsilon_L + \alpha_y \left| \mathbf{E}_y \right|^2, \tag{1b}$$

where ε_L is a frequency- dependent linear part,and α is a nonlinear coefficient.

Only TE modes are going to be considered with propagation constant k_x along the x-direction, and operating frequency ω. Electromagnetic fields are assumed to be

$$\mathbf{E} = \left[0,\ \mathbf{E}_y(z),\ 0\right] e^{i(k\ x - \omega\, t)} \tag{2}$$

$$\mathbf{H} = \left[\mathbf{H}_x, 0, \mathbf{H}_z\right] e^{i(k\ x - \omega\, t)} \tag{3}$$

Using Maxwell’s equations and following the usual derivation, then the electric and magnetic field components in each layer can be written as:

2.1. In a linear dielectric medium

$$E_y^{(1)} = E_o \exp[k_1 z] \tag{4}$$

$$H_x^{(1)} = \frac{iE_y^{(1)} k_1}{\omega\mu_o} \tag{5}$$

$$H_z^{(1)} = \frac{k_1}{\omega\mu_o} E_y^{(1)} \tag{6}$$

The electric and magnetic fields in the dielectric substrate are:

Where, $k_1{}^2 = k_x^2 - \frac{\omega}{c^2}^2 \varepsilon_1$, $\mu_o\varepsilon_o = \frac{1}{c^2}$, c is the velocity of light, and $k_x = \omega / c$, E_0 is the electric field in region 1 at z = 0

2.2 In nonlinear medium

The components of electric and magnetic fields are closely follow the treatment given by Boardman and Egan [(10)] as:

$$E_y^{(2)} = \mathrm{P\,cn}\,[\mathrm{q}(\mathrm{z} + \mathrm{z_o}) \mid m] \tag{7}$$

$$H_x^{(2)} = \frac{-iq}{\omega\mu_o} \mathrm{P\,sn}\,[\mathrm{q}(\mathrm{z} + \mathrm{z_o})]\,dn\,[\mathrm{q}(\mathrm{z} + \mathrm{z_o})] \tag{8}$$

$$H_z^{(2)} = \frac{k_2}{\omega\mu_o} [\mathrm{P\,cn}\,[\mathrm{q}(\mathrm{z} + \mathrm{z_o}) \mid m]] \tag{9}$$

Where $k_2^2 = k_x^2 - \frac{\omega^2 \varepsilon_2}{c^2}$, cn, sn and dn are specific Jacobian elliptic functions, and

$q = (k_2^4 + 4\Lambda_2 C_2)^{1/4}$, $\Lambda_2 = \frac{\omega^2 \alpha_2}{2c^2}$, and $p^2 = (q^2 + k_2^2)/2\Lambda_2$

C_2 is a constant that can be obtained from the boundary conditions.

2.3 In the ferrite cover

The electric and magnetic fields components in the ferrite cover can be written as [20]:

$$E_y{}^3 = E_d e^{k_3(d-z)} \tag{10}$$

$$\mathrm{H}_x^{(3)}(z) = \left(\frac{\mu_{xx} k_3 - i k_x \mu_{xz}}{i\omega\mu_o \mu_{xx}\mu_v}\right) E_y \tag{11}$$

$$H_z^{(3)}(z) = \left(\frac{\mu_{xz} k_3 + i k_x \mu_{xx}}{i\omega\mu_o \mu_{xx}\mu_v}\right) E_y \tag{12}$$

where E_d denotes the electric field in region (3) at z= d, $k_3 = (k_x^2 - \omega^2 \mu_v^2/c^2)$ and $\mu_v = (\mu_{xx}^2 + \mu_{xz}^2)/\mu_{xx}$, which is the effective permeability.

3. Dispersion Equations

Matching the field components H_x , H_z and E_y at the boundaries, z =

$cn[qd] =$

$$\frac{2\left(\frac{\alpha_2 E_o^2}{2}\right)\left(\frac{\alpha_2 E_d^2}{2}\right)\left\{Q^2 - n_1 \frac{c}{\omega}\left(\frac{k_3\mu_{xx} - ik\mu_{xz}}{\mu_{xx}\mu_v}\right)\right\}}{\frac{\alpha_2}{2} E_o E_d \left\{\frac{\alpha_2 E_o^2}{2} n_1^2 + \frac{\alpha_2 E_d^2}{2}\frac{c^2}{\omega^2}\left(\frac{k_3\mu_{xx} - ik_x\mu_{xz}}{\mu_{xx}\mu_v}\right) + \left(\frac{\alpha_2 E_o^2}{2} + \frac{\alpha_2 E_d^2}{2}\right)Q^2 + \left(\frac{\alpha_2 E_o^2}{2} - \frac{\alpha_2 E_d^2}{2}\right)\right\}} \tag{13}$$

0 and z = d, the dispersion equations are:

Where $n_1 = ck_1/\omega$, E_0 and E_d are the values of the electric field at the lower and upper boundaries of the film, respectively. This equation [13] is represented in the form

$f\,(n_x, \alpha E_0^2/2, \omega, D) = 0,$

where $n_x = ck_x/\omega$, $D = \omega d/c$, and $\alpha E_0^2/2$ is called the optical power density.

The conditions for the existence of nonlinear surface waves, for the present study are $n_x > \varepsilon_2, > \varepsilon_1$, $\mu_v < 0$ It is very clear that the dispersion

$$cn[qd] = \frac{2\left(\frac{\alpha_2 E_o^2}{2}\right)\left(\frac{\alpha_2 E_d^2}{2}\right)\left(Q^2 - n_1 n_3\right)}{\frac{\alpha_2 E_o\, Ed}{2}\left\{\left(n_1^2 \alpha_2\, E_o^2/2\right) + \left(n_3^2 \alpha_2\, E_d^2/2\right) + Q^2\left(\alpha_2\, E_o^2/2 + \alpha_2\, E_d^2/2\right) + \left(\frac{\alpha_2 E_o^2}{2} - \frac{\alpha_2 E_d^2}{2}\right)\right\}} \qquad (14)$$

equation displays non-reciprocal behavior which is very important in designing microwave devices since

$f\,(n_x, \alpha\, E_0^2/2, \omega, D) \neq f\,(-n_x, \alpha E_0^2/2, \omega, D)$

We examine the dispersion equation in the special case $\mu_{xz} = 0$ and $\mu_{xx} = 1$, we get,

$$\frac{\alpha_2^2}{(\mu_v\varepsilon_3 - \varepsilon_1)(\varepsilon_1 + \mu_v\varepsilon_3 - 2\varepsilon_2)}\left(E_d^2 - \frac{\varepsilon_3\mu_v - \varepsilon_2}{\alpha_2}\right)^2 - \frac{\alpha_2^2}{(\mu_v\varepsilon_3 - \varepsilon_1)(\varepsilon_1 + \mu_v\varepsilon_3 - 2\varepsilon_2)} \times \left(E_d{}^2 - \frac{\varepsilon_1 - \varepsilon_2}{\alpha_2}\right)^2 = 1 \qquad (15)$$

which is identical to the results obtained in [10].

After some algebraic manipulations, the relation between E_0^2 and E_d^2 can be derived as

4. Numerical Results, Discussions, and conclusions

The derived nonlinear characteristic dispersion equation Eq. (13) has been computed and simulated numerically. The results are presented in graphical form by using software mathematical program known as MAPLE V. In our computation, the data is used as reported by shabat [17], $\mu_o H_o = .5T$, $\mu_B = 1.25$, $\mu_o M_o = .175T$ and $\gamma = 1.76\times 10^{11}\,\mathrm{rad\,s^{-1}}T^{-1}$. For $\mu_v \leq 0$, the range of surface waves is from $\sqrt{f_o(f_o + f_m)}$ to $(f_o + f_m)$, where $f_0 = \omega_0/2\pi$, and

$f_m=\omega_m/2\pi$, This gives result of values corresponds to frequencies 16.234 GHz to 18.9 GHz.

Figure 2 shows dispersion curves as $\alpha E_0^2/2$ versus the effective wave index n_x of Eq. (13) in both wave propagation directions, since ($n_x = \pm 1$, $n_x = 1$ for propagation in the forward direction, $n_x = -1$ for the backward direction). Both curves represent the non-reciprocal behavior for different values of operating frequency. These values are very important in computing the power flow in the wave guide structures which are now under consideration. The figure illustrates the dependence of the wave index of TE_o waves on the optical power density at the lower boundary of the nonlinear layer. The curves are labeled with values of f: curves 1, 2, 3 represent different frequency as 16.40GHz, 16.43GHz and 16.46GHz respectively. In addition the results of the present work exhibit that the power density depends strongly on the frequencies, it might be also used in designing and implementing integrated microwave devices based on non-reciprocal behavior as isolators and switches. The curve of power density versus wave index decreased suddenly from high to lower point.

Figure 3 describes the relation between the power density ($\alpha E_0^2/2$) and the thickness of the film D for various values of frequency. We notice that the nonlinearly increases in strength as the film thickness increases in the forward and backward wave direction. Both figures also display non-reciprocal behavior.

We proposed here a new approach describing a non- reciprocal behavior of nonlinear surface waves in a three–layer ferrite wave guide structure, which is very promising for designing future microwave devices.

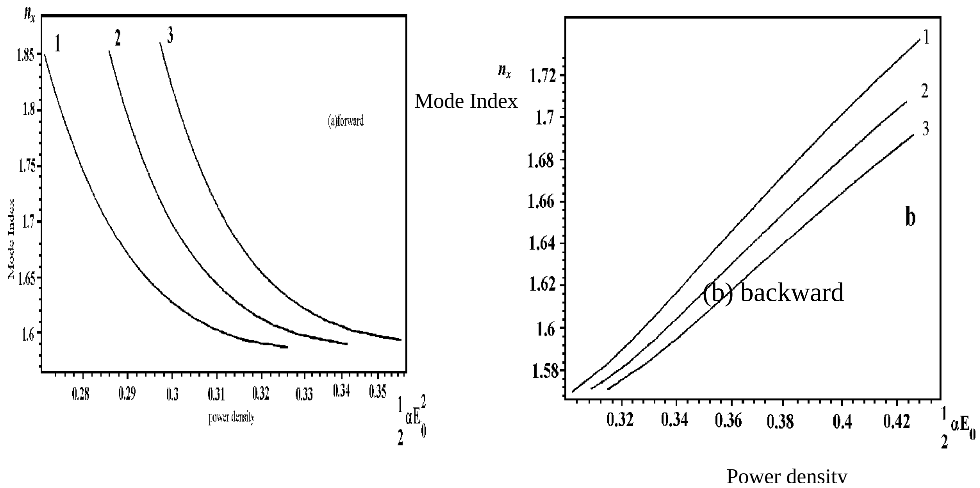
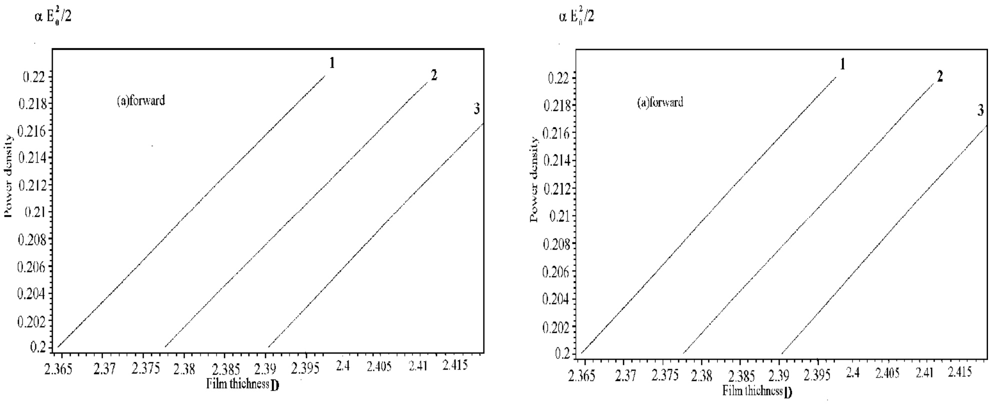


Figure (2): Computed power density versus the effective mode index for various values of frequency. (a) Forward, (b) backward (γ=1.67x10^{11} rad.$S^{-1}T^{-1}$, μ_0M_0=0.175T, μ_0H_0=0.5T,ε_f=1,

$\alpha E_d^2/2$=. 5,d=30μm). Curve (1), f=16.40GHz;curve (2), f=16.43GHz;curve (3), f=16.46GHz



Figure(3): Computed power density versus the film thickness for various values of the frequency, in two direction (a) forward (b) backward. curve1, f=16.40GHz; curve2, f=16.43GHz; curve3, f=16.46GHz

References

1) DB. Ostrowsky, and R. Reinish, "Guided Waves Nonlinear Optics", NATO Series E:**214**, Kluwer Academic Publishers, Dordecht, (1992).

2) NN. Akhmediev, "Novel Class of Nonlinear Surface Waves: Asymmetric Modes in a Symmetric Layered Structure", *Sov. Phys. Jetp*, **56**, (1982), 299-303.

3) F. Ledere, U. Langbein, and H.E. Ponath, "Nonlinear Waves Guided by a Dielectric Slab", *Appl. Phys*.-B, **31**, (1985), 69-73.

4) D. Mihalache, and H. Totia, "On TE- Polarized Nonliear Waves Guided by Dielectric Layered Structures", *Solid State Commun*, **45**, (1985), 175-177.

5) U. Langbein, F. Lederer, and H.E. Ponath, "Generalized Dispersion Relations for Nonlinear Slab Guided Waves", *Opt. Commun*, **53**, (1985), 417-420.

6) U. Langbein, F. Lederer, T. Peschel, and HE. Ponath, "Nonlinear Guided Waves in Saturable Nonlinear Media", *Opt. Lett*., **10**, (1985), 571.

7) D. Mihalache, D. Mazilu, and F. Ledere, "Nonlinear TE-Polarized Surface Plasmon Polaritons Guided by Metal Films", *Op. Commun*, **59**, (1986), 391-394.

8) GI. Stegeman, E.M. Wright, C.T. Seaton, J. Moloney, T. Shen, A.A. Maradudin, and R.F. Wallis, "Nonlinear Slab GuidedWaves in Non-Kerr-Like Media", *IEEE J. Quantum Electron*. QE-**32**, (1986), 977-983.

9) G.I. Stegman, C.T. Seaton, J. Ariyasu, T.P. Shen, and J.V. Moloney, "Saturation and Power Law Dependence of Nonlinear Waves Guided by a Single Interface", *Opt. Commun*, **56**, (1986), 365-368.

10) A.D. Boardman and P. Egan, "Optically Nonlinear Waves in Thin Films, IEEE of Quant. Electronics. QE-**22(2)**, February, (1986), 319.

11) A.D. Boardman, M.M. Shabat, and R.F. Wallis, "TE Waves at an Interface between Linear Gyromagnetic and Nonlinear Dielectric Media", *J. Phys. D.*, **24**, (1991), 1702-1707.

12) M.M. Shabat, and D. J ger, Nonlinear Electromagnetic Surface Waves Guided by a Single Hexagonal Planar Ferrite,"Proceedings of XIVth International Conference on Microwave Ferrite", Eger, Hugary (1985), 127-130.

13) M.M. Shabat, D. J ger, "Microwave Nonlinear Characteristics of TE Waves at Ferrite-Ferroelectric Interface,"Proceeding of 10th Microcoll", Budapest, Hungary, (1999), 343-346.

14) M.M. Shabat, "New Nonlinear Magnetostatic Surface Waves in a Metallised Ferromagnetic Film", *Opt. Applicata*, **43(4)**, (1994), 293-295.

15) Q. Wang, Z. Wu, S. Li., and L. Wang, “Nonlinear Magnetic Surface Waves On the Interface between Ferromagnet and antiferromagnet”, *J. App. Phy.*, **87(4)**, (2000), 1908-1913.

16) G.N. Burlak, V.V. Grimal’skii, and S.V. Koshevaya, “Nonlinear Waves In a Three Layer Ferroelectric-Ferrite- Ferroelectric System”, *Phy. Solid*, **35(8)**, (1993), 1049-1050.

17) M.M. Shabat, “Strongly Nonlinear Magnetostatic Surface Waves In a Ground Ferrite Film”, *phy. stat. sol.(a)*, **149**, (1995), 691-696.

18) I. Kasa, “Microwave Integrated Circuits”, Elsevier Publ. Co., Amsterdam, (1991).

19) M.S. Sodha and N. C. Srivastava, “Microwave Propagation in Ferrimagnetics”, Plenum Press, New York, (1981).

Appendix A

We consider a thin, optically nonlinear, dielectric film sandwiched between semi-infinit linear dielectrics. Only TE modes are going to be considered, and these propagate along the x axis with wave number kx and angular frequency ω. The electric and magnetic field component have the form $E= (0,E_y(z),0)\ exp(ik_x\text{-}i\omega t)$ and $H = (H_x(z),0,H_z(z))\ exp(ik_x\text{-}i\omega t)$. If the optical nonlinearity is of the kerr type, then the dielectric constant of the film is $\varepsilon_2+\ \alpha_2/E_2/^2$ so that if E_i are taken to be real, then they must satisfy the equation:

the first integration of equation (A.1) is derived as follows:

$$\ddot{E}_2-\left(k_2^2-2\Lambda_2E_2^2\right)E_2=o \qquad \text{(A.1)}$$

$$\frac{dE_2}{dz}\frac{d}{dz}\left(\frac{dE_2}{dz}\right)-k_2^2E_2\frac{dE_2}{dz}+2\Lambda E_2^3\frac{dE_2}{dz}=0$$

$$\frac{1}{2}\frac{d}{dz}\left(\frac{dE_2}{dz}\right)^2-\frac{1}{2}k_2^2\frac{dE_2^2}{dz}+2\Lambda.\frac{1}{4}\frac{dE_2^4}{dz}=0$$

$$\frac{1}{2}\left(\frac{dE_2}{dz}\right)^2-\frac{1}{2}k_2^2E_2^2+\frac{1}{2}\Lambda E_2^4=c_2$$

$$\left(\frac{dE_2}{dz}\right)^2-k_2^2E_2^2+\Lambda E_2^4=2.c_2 \qquad \text{(A.2)}$$

Then Eq. (A.2) gives:

$$\dot{E}_2^2 - \left(k_2^2 - \Lambda_2 E_2^2\right)E_2^2 = C_2 \qquad \text{(A.3)}$$

the value of constant C_2 is evaluated as:

$at \quad z = o$

$$E_1 = E_2 \;\;,\;\; E_1 = E_0 e^{o} = E_o \qquad \text{(A.4)}$$

$$,\;\; \dot{E}_1 = \dot{E}_2 \;\;,\; \dot{E}_1 = k_1 E_0 e^{o} = k_1 E_o \qquad \text{(A.5)}$$

$$\dot{E}_1^2 - \left(k_2^2 - \Lambda_2 E_1^2\right)E_1^2 = C_2$$

$$\dot{E}_0^2\, k_1 - \left(k_2^2 - \Lambda_2 E_0^2\right)E_0^2 = C_2$$

$$E_0^2\left(k_x^2 - \frac{\omega^2 \varepsilon_1}{c^2}\right) - \left(k_2^2 - \Lambda_2 E_0^2\right) E_0^2 = C_2 \qquad \text{(A.6)}$$

$$E_0^2 k_x^2 - \frac{\omega^2 \varepsilon_1}{c^2} E_0^2 - \left[\left(k_x^2 - \frac{\omega^2 \varepsilon_2}{c^2}\right) - \Lambda_2 E_0^2\right] E_0^2 = C_2$$

$$E_0^2 k_x^2 - \frac{\omega^2 \varepsilon_1}{c^2} E_0^2 - k_x^2 E_0^2 + \frac{\omega^2 \varepsilon_2}{c^2} E_0^2 + \Lambda_2 E_0^2\, E_0^2 = C_2 \qquad \text{(A.7)}$$

Finally we obtain:

$$\frac{\omega^2}{c^2} E_0^2\left[\varepsilon_2 - \varepsilon_1 + \frac{\alpha_2 E_0^2}{2}\right] = C_2 \qquad \text{At } z=0 \qquad \text{(A.8)}$$

$$\frac{\omega^2}{c^2} E_b^2\left[\varepsilon_2 - \varepsilon_3 + \frac{\alpha_2 E_b^2}{2}\right] = C_2 \qquad \text{At } z=d \qquad \text{(A.9)}$$

Where E_0^2, and E_b^2, are the value of electric fields at the lower and higher boundary, respectively.

Appendix B

The relationship between E_0^2 and E_b^2 is derived mathematically with details as follows:

Dividing both Eq's. (A.8) And (A.9), we get;

$$E_0^2\left(\varepsilon_2-\varepsilon_1+\alpha_2E_0^2/2\right)=E_b^2\left(\varepsilon_2-\varepsilon_3+\alpha_2E_b^2/2\right)$$

$$E_b^2\left(\varepsilon_2-\varepsilon_3+\alpha_2E_b^2/2\right)-E_o^2\left(\varepsilon_2-\varepsilon_1+\alpha_2E_o^2/2\right)=0 \qquad \text{(B.1)}$$

Multiply Eq. (B.1) by $2\alpha_2$:

$$2\alpha_2E_b^2\left(\varepsilon_2-\varepsilon_3+\alpha_2E_b^2/2\right)-2\alpha_2E_o^2\left(\varepsilon_2-\varepsilon_1+\alpha_2E_0^2/2\right)=0$$

$$2\alpha_2E_b^2\left(\varepsilon_2-\varepsilon_3\right)+\left(\alpha_2E_b^2\right)-2\alpha_2E_o^2\left(\varepsilon_2-\varepsilon_1\right)-\left(\alpha_2E_0^2\right)^2=0$$

$$\left(\alpha_2E_b^2\right)^2-\left(\alpha_2E_o^2\right)^2+\alpha_2E_b^2\left(2\varepsilon_2-2\varepsilon_3\right)+\alpha_2E_o^2\left(2\varepsilon_1-2\varepsilon_2\right)=0 \qquad \text{(B.2)}$$

$$\left(\alpha_2E_b^2\right)^2-\left(\alpha_2E_o^2\right)^2-\alpha_2E_b^2\left(\varepsilon_1+\varepsilon_3-2\varepsilon_2+\varepsilon_3-\varepsilon_1\right)+$$
$$\alpha_2E_0^2\left(\varepsilon_1+\varepsilon_3-2\varepsilon_2-\varepsilon_3+\varepsilon_1\right)=0 \qquad \text{(B.3)}$$

Then, add $\left(\varepsilon_3-\varepsilon_1\right)\left(\varepsilon_1+\varepsilon_3-2\varepsilon_2\right)$ to both sides of Eq. (B.3), we have:

$$\left(\alpha_2E_b^2\right)^2-\left(\alpha_2E_o^2\right)^2-\alpha_2E_b^2\left(\varepsilon_1+\varepsilon_3-2\varepsilon_3+\varepsilon_3-\varepsilon_1\right)+\alpha_2E_0^2$$

$$\left(\varepsilon_1+\varepsilon_3-2\varepsilon_2-\varepsilon_3+\varepsilon_1\right)+\left(\varepsilon_3-\varepsilon_1\right)\left(\varepsilon_1+\varepsilon_3-2\varepsilon_2\right)=\left(\varepsilon_3-\varepsilon_1\right)\left(\varepsilon_1+\varepsilon_3-2\varepsilon_2\right) \qquad \text{(B.4)}$$

Adding the term, $\left(\alpha_2E_b^2\alpha_2E_o^2-\alpha_2E_b^2\alpha_2E_o^2\right)$ in left-hand side of Eq. (B.3) we got;

$$\left(\alpha_2 E_b^2\right)^2 + \alpha_2 E_b^2 \alpha_2 E_o^2 - \alpha_2 E_b^2 \alpha_2 E_o^2 - \left(\alpha_2 E_o^2\right)^2 - \alpha_2 E_b^2\left(\varepsilon_1 + \varepsilon_3 - 2\varepsilon_2\right)$$

$$- \alpha_2 E_b^2\left(\varepsilon_3 - \varepsilon_1\right) + \alpha_2 E_o^2\left(\varepsilon_1 - \varepsilon_3 - 2\varepsilon_2\right) - \alpha_2 E_o^2\left(\varepsilon_3 - \varepsilon_1\right)$$

$$+ \left(\varepsilon_3 - \varepsilon_1\right)\left(\varepsilon_1 + \varepsilon_3 - 2\varepsilon_2\right) = \left(\varepsilon_3 - \varepsilon_1\right)\left(\varepsilon_1 + \varepsilon_3 - 2\varepsilon_2\right) \tag{B.5}$$

$$\left[\alpha_2 E_b^2 - \alpha_2 E_o^2 - \left(\varepsilon_3 - \varepsilon_1\right)\right]\left[\alpha_2 E_b^2 + \alpha_2 E_o^2 - \left(\varepsilon_1 + \varepsilon_3 - 2\varepsilon_2\right)\right] = \left(\varepsilon_3 - \varepsilon_1\right)\left(\varepsilon_1 + \varepsilon_3 - 2\varepsilon_2\right) \tag{B.6}$$

$$\left[\left(\alpha_2 E_b^2 - \varepsilon_3 + \varepsilon_2\right) - \alpha E_o^2 + \varepsilon_1 - \varepsilon_2\right]\left[\alpha_2 E_b^2 - \varepsilon_3 + \varepsilon_2 + \alpha_2 E_o^2 - \varepsilon_1 + \varepsilon_2\right] \tag{B.7}$$
$$= \left(\varepsilon_3 - \varepsilon_1\right)\left(\varepsilon_1 + \varepsilon_3 - 2\varepsilon_2\right)$$

$$\left(\alpha_2 E_b^2 - \varepsilon_3 + \varepsilon_2\right)^2 - \left(\alpha E_o^2 - \varepsilon_1 + \varepsilon_2\right)^2 = \left(\varepsilon_3 - \varepsilon_1\right)\left(\varepsilon_1 + \varepsilon_3 - 2\varepsilon_2\right) \tag{B.8}$$

$$\alpha_2^2\left[E_b^2 - \frac{\varepsilon_3 - \varepsilon_2}{\alpha_2}\right]^2 - \alpha_2^2\left[E_o^2 - \frac{\varepsilon_1 - \varepsilon_2}{\alpha_2}\right]^2 = \left(\varepsilon_3 - \varepsilon_1\right)\left(\varepsilon_1 + \varepsilon_3 - 2\varepsilon_2\right) \tag{B.9}$$

At last, we get the relation between E_0^2 and E_b^2 as:

$$\frac{\alpha_2^2}{\left(\varepsilon_3 - \varepsilon_1\right)\left(\varepsilon_1 + \varepsilon_3 - 2\varepsilon_2\right)}\left(E_b^2 - \frac{\varepsilon_3 - \varepsilon_2}{\alpha_2}\right)^2 - \frac{\alpha_2^2}{\left(\varepsilon_3 - \varepsilon_1\right)\left(\varepsilon_1 + \varepsilon_3 - 2\varepsilon_2\right)} \times$$
$$\times \left(E_o^2 - \frac{\varepsilon_1 - \varepsilon_2}{\alpha_2}\right)^2 = 1 \tag{B.10}$$

Appendix C

$$\frac{E_o}{P} = cn(z_o q)$$

$$\frac{E_o^2}{P^2} = cn^2(z_o q)$$

$$1 - \frac{E_o^2}{P^2} = 1 - cn^2(z_o q)$$

$$,\ 1 - \frac{E_o^2}{P^2} = sn^2(z_o q)$$

The derivation of the constant C_2 in both boundaries:

(1) At z = o

From Eq. (3.46) we have;

$$\dot{E}_2^2 - \left(k_2^2 - \Lambda_2 E_2^2\right)E_2^2 = C_2$$

But $\quad E_y^{(1)} = E_o \exp(k_1 z)$

And $\quad \dot{E}_1 = E_1 k_1\ ,\ \dot{E}_1^2 = E_1^2 k_1^2 \qquad$ (C.1)

Where $\quad E_y^{(1)} \equiv E_1 \quad$ and $\quad E_y^{(2)} \equiv E_y$

And $\quad E_1 = E_2$ and $\dot{E}_1 = \dot{E}_2$

Then

$$E_1^2 k_1^2 - \left(k_2^2 - \Lambda_2 E_2^2\right)E_2^2 = C_2$$

$$E_o^2\left[k^2 - \frac{\omega^2 \varepsilon_1}{c^2}\right] - \left(k_2^2 - \Lambda E_2^2\right)E_2^2 = C_2 \qquad \text{(C.2)}$$

Where $k_1^2 = k^2 - \frac{\omega^2 \varepsilon_1}{c^2}$

And $k_2^2 = k^2 - \frac{\omega^2 \varepsilon_2}{c^2}$

Equation (4.42) becomes:

$$\frac{\omega^2 E_o^2}{c^2}\left[\varepsilon_2 - \varepsilon_1 + \frac{\alpha_2 E_o^2}{2}\right] = C_2 \qquad \text{(C.3)}$$

Where E_o is the value of electric field at the lower boundary of the film

(2) At z = d

We have $E_y^{(3)} = E_d \exp\left[k_3(d - z)\right]$

Then $\dot{E}_3 = -k_3 E_3$

Where $E_y^{(3)} \equiv E_3$

And $\left(\dot{E}_3\right)^2 = k_3^2 E_3^2$

Substitute in:

$$\dot{E}_2^2 - \left(k_2{}^2 - \Lambda_2 E_2{}^2\right) E_2{}^2 = C_2 \qquad \text{(C.4)}$$

But at $\mathrm{z} = \mathrm{d}$ $E_2 = E_3$, $\dot{E}_2 = \dot{E}_3$

And $\mathrm{k}_2^2 = k^2 - \frac{\omega^2 \varepsilon_2}{c^2}$, $\mathrm{k}_3^2 = k^2 - \frac{\omega^2}{c^2} \varepsilon_3 \mu_v$

Then Eq. (4.44) becomes:

$$k_3^2 E_3^2 - \left[\left(k^2 - \frac{\omega^2}{c^2}\varepsilon_2\right)\Lambda_2 E_2^2\right] E_2^2 = C_2$$

$$E_d^2\left(k^2 - \frac{\omega^2}{c^2}\varepsilon_f\,\mu_v\right) - \left[\left(k^2 - \frac{\omega^2}{c^2}\varepsilon_2\right)\Lambda_2 E_d^2\right] E_d^2 = C_2 \qquad \text{(C.5)}$$

Then

$$\frac{\omega^2}{c^2} E_d^2 - \left[\varepsilon_2 - \varepsilon_3\,\mu_v + \frac{\alpha_2}{2}\,E_d^2\right] = C_2, \qquad \text{(C.6)}$$

Relationship between E_0^2 and E_d^2 :

We drive mathematically the relationship between E_0^2 and E_d^2 as follow:

Dividing Eq. (C.3) by Eq. (C.4) we get:

$$E_d^2\left(\varepsilon_2 - \varepsilon_3\mu_v + \frac{\alpha_2}{2}E_d^2\right) = E_0^2\left(\varepsilon_2 - \varepsilon_1 + \frac{1}{2}\alpha_2 E_0^2\right) \qquad \text{(C.7)}$$

Multiply Eq. (C.7) by 2α, it becomes:

$$2\alpha E_d^2\left(\varepsilon_2 - \varepsilon_3\mu_v + \frac{1}{2}\alpha E_d^2\right) - 2\alpha E_0^2\left(\varepsilon_2 - \varepsilon_1 + \frac{1}{2}\alpha_2 E_0^2\right) = 0 \qquad \text{(C.8)}$$

$$2\alpha E_d^2\left(\varepsilon_2 - \varepsilon_3\mu_v\right) + \left(\alpha E_d^2\right)^2 - 2\alpha E_0^2\left(\varepsilon_2 - \varepsilon_1\right) - \left(\alpha_2 E_0^2\right)^2 = 0$$

$$\left(\alpha_2 E_d^2\right)^2 - \left(\alpha_2 E_0^2\right)^2 + \alpha_2 E_d^2\left(2\varepsilon_2 - 2\varepsilon_3\mu_v\right) + \alpha_2 E_0^2\left(2\varepsilon_1 - 2\varepsilon_2\right) = 0$$

$$\left(\alpha_2 E_d^2\right)^2-\left(\alpha_2 E_0^2\right)^2-\alpha_2 E_d^2\left(\varepsilon_1+2\varepsilon_3\mu_v-2\varepsilon_2-\varepsilon_1\right)+\alpha_2 E_0^2\left(\varepsilon_1+\varepsilon_1+\mu_v\varepsilon_3-\mu_v\varepsilon_3-2\varepsilon_2\right)=0$$

Adding $\left(\mu_v\varepsilon_3-\varepsilon_1\right)\left(\varepsilon_1+\mu_v\varepsilon_3-2\varepsilon_2\right)$ to the both sides of Eq. (C.7) then the left-hand side of Eq. (C.7) becomes:

$$\left(\alpha_2 E_d^2\right)^2-\left(\alpha_2 E_o^2\right)^2-\alpha_2 E_d^2\left(\varepsilon_1+\mu_v\varepsilon_3-2\varepsilon_2+\mu_v\varepsilon_3-\varepsilon_1\right)+\alpha_2 E_o^2$$
$$\left(\varepsilon_1+\mu_v\varepsilon_3-2\varepsilon_2-\mu_v\varepsilon_3+\varepsilon_1\right)+\left(\mu_v\varepsilon_3-\varepsilon_1\right)\left(\varepsilon_1+\mu_v\varepsilon_3-2\varepsilon_2\right)$$

Also adding the term $\alpha_2 E_d^2\alpha_2 E_o^2-\alpha_2 E_d^2\alpha_2 E_o^2$ to the left-hand side, we obtain:

$$\left(\alpha_2 E_d^2\right)^2+\alpha_2 E_d^2\alpha_2 E_o^2-\alpha_2 E_d^2\alpha_2 E_o^2-\left(\alpha_2 E_o^2\right)^2-\alpha_2 E_d^2\left(\varepsilon_1+\mu_v\varepsilon_3-2\varepsilon_2\right)-\alpha_2 E_d^2$$
$$\left(\varepsilon_3\mu_v-\varepsilon_1\right)+\alpha_2 E_o^2\left(\varepsilon_1+\mu_v\varepsilon_3-2\varepsilon_2\right)-\alpha_2 E_o^2\left(\varepsilon_3\mu_v-\varepsilon_1\right)+\left(\mu_v\varepsilon_3-\varepsilon_1\right)\left(\varepsilon_1+\mu_v\varepsilon_3-2\varepsilon_2\right)$$

The above equation becomes:

$$\left[\alpha_2 E_d^2-\alpha_2 E_o^2-\left(\mu_v\varepsilon_3-\varepsilon_1\right)\right]\left[\alpha_2 E_d^2+\alpha_2 E_o^2-\left(\varepsilon_1+\mu_v\varepsilon_3-2\varepsilon_1\right)\right]$$

Finally the Eq. (C.9) lead to:

$$\left[\alpha_2 E_d^2-\alpha_2 E_o^2-\left(\mu_v\varepsilon_3-\varepsilon_1\right)\right]\left[\alpha_2 E_d^2+\alpha_2 E_o^2-\left(\varepsilon_1+\mu_v\varepsilon_3-2\varepsilon_2\right)\right]=\left(\mu_v\varepsilon_3-\varepsilon_1\right)\left(\varepsilon_1+\mu_v\varepsilon_3-2\varepsilon_2\right) \quad \text{(C.9)}$$

$$\left[\alpha_2 E_d^2-\alpha_2 E_o^2+\varepsilon_1\mu_v\varepsilon_3-\varepsilon_2+\varepsilon_2\right]\left[\alpha_2 E_d^2-\varepsilon_3\mu_v-\varepsilon_1+\varepsilon_3+\varepsilon_2+\alpha_2 E_o^2\right]$$
$$=\left(\mu_v\varepsilon_3-\varepsilon_1\right)\left(\varepsilon_1+\mu_v\varepsilon_3-2\varepsilon_2\right)$$

$$\left[\left(\alpha_2 E_d^2-\varepsilon_3\mu_v+\varepsilon_3\right)-\left(\alpha_2 E_o^2-\varepsilon_1+\varepsilon_2\right)\right]\left[\left(\alpha_2 E_d^2-\varepsilon_3\mu_v+\varepsilon_3\right)-\left(\alpha_2 E_o^2-\varepsilon_1+\varepsilon_2\right)\right] \quad \text{(C.10)}$$
$$=\left(\mu_v\varepsilon_3-\varepsilon_1\right)\left(\varepsilon_1+\mu_v\varepsilon_3-2\varepsilon_2\right)$$

, Equation (C.10) becomes:

$$\left(\alpha_2 E_d^2 - \varepsilon_3\mu_v + \varepsilon_3\right)^2 - \left(\alpha_2 E_o^2 - \varepsilon_1 + \varepsilon_2\right)^2 = \left(\mu_v\varepsilon_3 - \varepsilon_1\right)\left(\varepsilon_1 + \mu_v\varepsilon_3 - 2\varepsilon_2\right) \qquad \text{(C.11)}$$

$$\alpha_2^2\left[E_d^2 - \frac{\varepsilon_3\mu_v - \varepsilon_2}{\alpha_2}\right]^2 - \alpha_2^2\left[E_o^2 - \frac{\varepsilon_1 - \varepsilon_2}{\alpha_2}\right]^2 = \left(\mu_v\varepsilon_3 - \varepsilon_1\right)\left(\varepsilon_1 + \mu_v\varepsilon_3 - 2\varepsilon_2\right)$$

Dividing Eq. (C.11) by $\left(\mu_v\varepsilon_3 - \varepsilon_1\right)\left(\varepsilon_1 + \mu_v\varepsilon_3 - 2\varepsilon_2\right)$, we obtain:

$$\frac{\alpha_2^2}{\left(\mu_v\varepsilon_3 - \varepsilon_1\right)\left(\varepsilon_1 + \mu_v\varepsilon_3 - 2\varepsilon_2\right)}\left(E_b^2 - \frac{\varepsilon_3\mu_v - \varepsilon_2}{\alpha_2}\right)^2 - \frac{\alpha_2^2}{\left(\mu_v\varepsilon_3 - \varepsilon_1\right)\left(\varepsilon_1 + \mu_v\varepsilon_3 - 2\varepsilon_2\right)} \times$$

$$\times\left(E_o^2 - \frac{\varepsilon_1 - \varepsilon_2}{\alpha_2}\right)^2 = 1 \qquad \text{(C.12)}$$

By using Eq. (C.12) we plot the relations between $\alpha_2 E_o^2$ against $\alpha_2 E_d^2$.

This (E_0^2, E_d^2) relationship can be used to great effect in the nonlinear generalisation of the dispersion relationships of asymmetric and symmetric waveguides that are familiar from linear solid state optics.

To find Eq(13) mahematically

Applying the boundary conditions on *E* and *H* at both z = 0 and z=d we obtain:

$$E_y^{(1)} = E_y^{(2)}$$

then
$$E_o \exp(k_1 z) = \mathrm{P}\,\mathrm{cn}\left[q(z_o)\middle| m\right]$$
$$E_o = \mathrm{P}\, cn(z_o q) \quad , \tag{1}$$
$$H_x^{(1)} = H_x^{(2)}$$

then
$$\frac{iE_y^{(1)} k_1}{\omega\mu_o} = \frac{-i}{\omega\mu_o} \mathrm{P}q\, \mathrm{sn}\,[\mathrm{qz_o}]dn[qz_o]$$
$$\mathrm{sn}\,[\mathrm{qz_o}]dn[qz_o] = \frac{-E_o k_1}{\mathrm{P}q} \tag{2}$$

and
$$E_y^{(2)} = E_y^{(3)}$$
$$\mathrm{P}\ \mathrm{cn}\ [\mathrm{q}(\mathrm{d} + \mathrm{z_o})] = E_d \exp\ [\mathrm{k}_3(d - d)]$$

then
$$E_d = Pcn\left\{q\left[d + \frac{1}{q} cn^{-1}(E_o / P)\right]\right\} \tag{3}$$

and

$$q = \left(k_2^4 + 4 \wedge_2 C_2\right)^{1/4} \quad , \quad \Lambda_2 = \frac{\omega^2 \alpha_2}{2c^2}$$
$$P^2 = \left(q^2 + k_2^2\right)/2\Lambda_2$$

From Eq's. (4.19) And (4.21) we have:

$$\frac{E_o}{P} = cn\ (z_o q) \tag{4}$$
$$,\ 1 - \frac{E_o^2}{P^2} = sn^2\ (z_o q) \tag{5}$$

where: $cn^2+sn^2=1$

Similarly; for the upper boundary, we have:

$$\frac{E_d}{P} = cn\left[q\left(d + \frac{1}{q}cn^{-1}(E_o/P)\right)\right]$$

$$, \quad 1 - \frac{E_d^2}{P^2} = sn^2\left[q\left(d + \frac{1}{q}cn^{-1}(E_o/P)\right)\right] \tag{6}$$

The boundary condition for the magnetic field at z = d is

$$H_x^{(2)} = H_x^{(3)}$$

$$\frac{-i}{\omega\mu_o}Pq\,\mathrm{sn}\left[\mathrm{q}(\mathrm{d}+\mathrm{z_o})\right]dn\left[q(d+z_o)\right] = \left(\frac{\mu_{xx}k_3 - ik\mu_{xz}}{i\omega\mu_o\mu_{xx}\mu_v}\right)E_d$$

$$E_d = \frac{\mu_{xx}\mu_v}{\mu_{xx}k_3 - ik\mu_{xz}}\left\{Pq\,\mathrm{sn}\left[\mathrm{q}(\mathrm{d}+\mathrm{z_o})\right]dn\left[q(d+z_o)\right]\right\} \tag{7}$$

and

$$B_z^{(3)} = B_z^{(2)}$$

$$-\mu_{xz}H_x^{(3)} + \mu_{xx}H_z^{(3)} = H_z^{(2)}$$

where $B = \mu(\omega)H$ (the magnetic flux density) .

Substitute about $H_z^{(2)}$, $H_z^{(3)}$ and $H_x^{(3)}$ we get:

then $$-\mu_{xz}\left[\frac{\mu_{xx}k_3 - ik\mu_{x2}}{i\omega\mu_o\mu_{xx}\mu_v}\right]E_d + \mu_{xx}\left[\frac{\mu_{xz}k_3 + ik\mu_{xx}}{i\omega\mu_o\mu_{xx}\mu_v}\right]E_d = \frac{k}{\omega\mu_o}\left[Pcn[q(d+z_o)]\right] \tag{8}$$

let

$$E_\mathrm{d} = Pcn[q(d+z_o)]$$

$$\mathrm{cn}\,[\mathrm{qd}] = \mathrm{cn}\,[\mathrm{qz_o} - qd - qz_o]$$

Using Jacobian Functions for Eq. (4.28) we get [12]:

$$\mathrm{cn}[qd]=\frac{\mathrm{cn}[qz_o]\,\mathrm{cn}[q(d+z_o)+\mathrm{sn}[qz_o]\,\mathrm{dn}[qz_o]\,\mathrm{sn}[q(d+z_o)]\,\mathrm{dn}[(d+z_o)]q}{1-\mathrm{m}\ \mathrm{sn}^2[qz_o]\ \mathrm{sn}^2[q(d+z_o)]} \quad (9)$$

Substitute from Eq's. (4.20), (4.22), (4.23), (4.24) and (4.25) in Eq. (4.29) we obtain:

$$cn[qd]=\frac{E_o}{P}\frac{E_d}{P}+\left(\frac{-E_ok_1}{Pq}\left\{\frac{\mu_{xx}k_3-ik\mu_{xz}}{\mu_{xx}\mu_v}\right\}\frac{E_d}{Pq}\right)\Big/1-m\left(1-\frac{E_o^2}{P^2}\right)\left(1-\frac{E_d^2}{P^2}\right)$$

$$cn[qd]=\frac{E_oE_d}{P^2}-\frac{E_oE_dk_1}{P^2q^2}\left\{\frac{\mu_{xx}k_3-ik\mu_{xz}}{\mu_{xx}\mu_v}\right\}\Big/1-m\left(1-\frac{E_o^2}{P^2}\right)\left(1-\frac{E_d^2}{P^2}\right)$$

$$cn[qd]=\frac{E_oE_d}{P^2}\left[1-\frac{k_1}{q^2}\left\{\frac{\mu_{xx}k_3-ik\mu_{xz}}{\mu_{xx}\mu_v}\right\}\right]\Big/1-m\left(1-\frac{E_o^2}{P^2}\right)\left(1-\frac{E_d^2}{P^2}\right)$$

$$cn[qd]=2E_oE_d\left[q^2-k_1\left\{\frac{\mu_{xx}k_3-ik\mu_{xz}}{\mu_{xx}\mu_v}\right\}\right]\Big/2q^2P^2\left\{1-m\left(1-\frac{E_o^2}{P^2}\right)\left(1-\frac{E_d^2}{P^2}\right)\right\} \quad (10)$$

then Eq. (4.30) becomes:

$$cn[qd]=2E_oE_d\left[q^2-k_1\left\{\frac{\mu_{xx}k_3-ik\mu_{xz}}{\mu_{xx}\mu_v}\right\}\right]\Big/k_1^2E_0^2+\left\{\frac{\mu_{xx}k_3-ik\mu_{xz}}{\mu_{xx}\mu_v}\right\}^2 \times E_d^2+q^2\left(E_o^2+E_b^2\right)+\Lambda_2\left(E_o^2+E_b^2\right)^2 \quad (11)$$

Multiply the right hand side of Eq. (4.33) by

$$\frac{c^2}{\omega^2}\left(\frac{\alpha_2}{2}\right)^2 E_o E_d \Big/ \frac{c^2}{\omega^2}\left(\frac{\alpha_2}{2}\right)^2 E_o E_d$$, We get:

$cn[qd] =$

$$\frac{2\left(\frac{\alpha_2 E_o^2}{2}\right)\left(\frac{\alpha_2 E_d^2}{2}\right)\left\{Q^2 - n_1 \frac{c}{\omega}\left(\frac{k_3\mu_{xx} - ik\mu_{xz}}{\mu_{xx}\mu_v}\right)\right\}}{\frac{\alpha_2}{2}E_o E_d\left\{\frac{\alpha_2 E_o^2}{2} n_1^2 + \frac{\alpha_2 E_d^2}{2}\frac{c^2}{\omega^2}\left(\frac{k_3\mu_{xx} - ik\mu_{xz}}{\mu_{xx}\mu_v}\right) + \left(\frac{\alpha_2 E_o^2}{2} + \frac{\alpha_2 E_d^2}{2}\right)Q^2 + \left(\frac{\alpha_2 E_o^2}{2} - \frac{\alpha_2 E_d^2}{2}\right)\right\}} \quad (12)$$